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A Hardy Space Associated with Twisted Convolution*

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INTRODUCTION

Consider on $\mathbf{C}^n$ the $2n$ linear differential operators

$$Z_j = \frac{\partial}{\partial z_j} + \bar{z}_j, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - z_j, \qquad j = 1,..., n.$$

They generate a nilpotent Lie algebra isomorphic with the Heisenberg algebra of dimension $2n + 1$, where the only non-trivial commutation rules are

$$[Z_j, \bar{Z}_j] = -2I.$$

These operators generate a family of "twisted translations" τ_w on $\mathbf{C}^n$ defined on measurable functions by

$$\begin{aligned}(\tau_w f)(z) &= \exp\left(\tfrac{1}{2} \sum (w_j Z_j + \bar{w}_j \bar{Z}_j)\right) f(z) \\ &= f(z + w) \exp(i \operatorname{Im}(z \cdot \bar{w})) \qquad (w \in \mathbf{C}^n).\end{aligned}$$

The mapping τ is the projective regular representation of $\mathbf{C}^n$ corresponding

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to the cocycle $\omega(z, w) = \exp(i \operatorname{Im}(z \cdot \bar{w}))$ [9]. The "twisted convolution" of two functions f and g on $\mathbf{C}^n$ can now be defined as

$$(f \times g)(z) = \int_{\mathbf{C}^n} f(w)\, \tau_{-w} g(z)\, dw$$

$$= \int_{\mathbf{C}^n} f(z-w)\, g(w)\, \bar{\omega}(z, w)\, dw.$$

Twisted convolution has been investigated in connection with the Weyl functional calculus by several authors [1, 6, 14].

More recently, L^p-boundedness of twisted convolution operators has been studied by one of us [10], who proved a multiplier theorem of Hörmander type.

The purpose of this paper is to define and describe a Hardy space $\mathscr{H}^1$ naturally associated with twisted convolution operators.

In order to define this space, we consider certain maximal operators in terms of twisted convolutions. The Hardy space will then consist of those distributions which are mapped into $L^1(\mathbf{C}^n)$ by any one of these maximal operators.

Some of the maximal operators we consider are the analogues in the twisted case of those which define H^1 in the theory of Fefferman and Stein [3].

Others correspond to the maximal operators which define the local Hardy space h^1 studied by Goldberg [5].

It is a remarkable difference from the classical case that both kinds of operators define the same twisted Hardy space. $\mathscr{H}^1$ can also be identified with a closed subspace of the Hardy space on a Heisenberg group with compact center. By exploiting this identification other properties of $\mathscr{H}^1$ can be derived. On the other hand we are able to obtain a characterization of $\mathscr{H}^1$ in terms of Riesz transforms, while no similar characterization is known for the Heisenberg group.

We also study the boundedness on $\mathscr{H}^1$ of twisted convolution operators defined by singular kernels.

Under the ordinary regularity assumptions, operators defined by kernels which are singular only at the origin are shown to be bounded on $\mathscr{H}^1$. Such operators naturally arise when dealing with potentials associated with the differential operator

$$L = \sum_j (Z_j \bar{Z}_j + \bar{Z}_j Z_j).$$

Singularities at infinity behave in a different way: in general, the same regularity assumptions only yield boundedness from $\mathscr{H}^1$ to L^1.

1. Statement of the Main Results

We start by introducing the basic notations (see also [3, 9, 10, 13]).

Let $\mathscr{B}$ denote the class of C^∞-functions φ on $\mathbf{C}^n$, supported on the cube $Q(0, 1)$ of center zero and half-side one, such that $\|\varphi\|_\infty \leqslant 1$ and $\|\nabla\varphi\|_\infty \leqslant 2$.

For $t > 0$, let $\varphi_t(z) = t^{-2n}\varphi(z/t)$.

Given σ, $0 < \sigma \leqslant +\infty$, and a tempered distribution f, define the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi \in \mathscr{B}} \sup_{0<t<\sigma} |\varphi_t \times f(z)|.$$

The definition of atom introduced by Coifman and Weiss [2] will be adapted as follows: an atom for $\mathscr{H}^1$ centered at $z \in \mathbf{C}^n$ is a function a supported in the cube $Q(z, r)$ of center z and half-side r, such that $\|a\|_\infty \leqslant (2r)^{-2n}$ and $\int a(w)\,\bar{\omega}(z, w)\,dw = 0$.

Let ψ be a C^∞-function on $\mathbf{C}^n$ with compact support and such that $\psi \equiv 1$ on a neighborhood of zero. Define

$$R_j(z) = \frac{z_j}{|z|^{2n+1}}\,\psi(z), \qquad \bar{R}_j(z) = \frac{\bar{z}_j}{|z|^{2n+1}}\,\psi(z) \tag{1.1}$$

for $j = 1, \ldots, n$.

We refer to the singular integral operators $\mathbf{R}_j$, $\bar{\mathbf{R}}_j$ defined by left twisted convolution with these kernels as the Riesz transforms. The terminology is motivated by the fact that they are essentially the operators formally defined as $Z_j L^{-1/2}$, $\bar{Z}_j L^{-1/2}$, $j = 1, \ldots, n$.

THEOREM A. *For a tempered distribution f on $\mathbf{C}^n$ the following are equivalent*:

(i) *for some σ, $0 < \sigma < +\infty$, $M_\sigma f \in L^1$.*

(ii) $M_\infty f \in L^1$.

(iii) *For some radial $\varphi \in \mathscr{S}$, such that $\int \varphi(z)\,dz \neq 0$,*

$$\sup_{0<t<1} |\varphi_t \times f(z)| \in L^1.$$

(iv) *f can be decomposed as $f(z) = \sum_j \lambda_j a_j(z)$, where the a_j's are atoms supported on cubes $Q(z_j, r_j)$, $r_j \leqslant 2\sqrt{\pi}$ and $\sum_j |\lambda_j| < +\infty$.*

(v) *$f \in L^1$ and $R_j \times f$, $\bar{R}_j \times f \in L^1$, for $j = 1, \ldots, n$.*

(vi) *$f(z)\exp(-i\theta)$ belongs to the Hardy space $H^1(\mathbf{K}_n)$, where $\mathbf{K}_n$ denotes the $2n + 1$-dimensional Heisenberg group with compact center.*

References and details on the space $H^1(\mathbf{K}_n)$ will be given in Section 3.

DEFINITION. The space of functions f satisfying any of the conditions in Theorem A will be called twisted Hardy space and denoted by $\mathscr{H}^1$. Equivalent norms for f in $\mathscr{H}^1$ are

(a) the L^1-norm of any of the maximal functions in (i)–(iii),

(b) $\inf\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j$ as in (iv)$\}$,

(c) $\|f\|_1 + \sum_{1 < j < n}(\|R_j \times f\|_1 + \|\bar{R}_j \times f\|_1)$,

(d) the norm of $f(z) \exp(-i\theta)$ in $H^1(\mathbf{K}_n)$.

Define $\mathscr{BMO}$ as the space of locally integrable functions on $\mathbf{C}^n$ such that for every cube $Q = Q(z, r)$, $r \leqslant 2\sqrt{\pi}$ and some constant $K > 0$

$$\frac{1}{|Q|}\int_Q \left| b(w) - \left(\frac{1}{|Q|}\int_Q b(u)\,\bar{\omega}(z, u)\,du\right)\omega(z, w)\right| dw \leqslant K. \tag{1.2}$$

The norm $\|b\|_*$ of b in $\mathscr{BMO}$ is the least value of K for which (1.2) holds.

THEOREM B. (1) *$\mathscr{BMO}$ is the dual of $\mathscr{H}^1$ in the following sense:*

(a) *suppose $b \in \mathscr{BMO}$. Then the mapping $a \to \int a(w)\,\bar{b}(w)\,dw$ initially defined on atoms, extends to a bounded linear functional on $\mathscr{H}^1$.*

(b) *Conversely, if Φ is a continuous linear functional on $\mathscr{H}^1$, there is a unique $b \in \mathscr{BMO}$ such that $\Phi(a) = \int a(w)\,\bar{b}(w)\,dw$ for every atom a. The norm of Φ is equivalent to $\|b\|_*$.*

(2) *A function b in $\mathscr{BMO}$ satisfies:*

(c) *if $Q = Q(z, r)$ and $r \leqslant 2\sqrt{\pi}$, $b(w)\,\bar{\omega}(z, w) \in BMO(Q)$ with norm independent of Q;*

(d) *if $Q = Q(z, r)$, $r > r_0 > 0$, then $1/|Q| \int_Q |b(w)|\,dw \leqslant c(r_0)\,\|b\|_*$.*

In particular, condition (1.2) *holds without restriction on the size of Q.*

The results on singular integrals are the object of the next two theorems.

THEOREM C. *Let K be a function with compact support such that*

$$\int_{|z| > 2|w|} |K(z - w) - K(z)|\,dz \leqslant A, \tag{1.3}$$

$$\text{either } \|K \times f\|_2 \leqslant B\,\|f\|_2 \text{ or } |\hat{K}(\zeta)| \leqslant B. \tag{1.4}$$

Then $\mathbf{K}f = K \times f$ is a bounded operator of $\mathscr{H}^1$ into itself.

THEOREM D. *If K is a function which vanishes on a neighborhood of zero and satisfies*

$$\int_{|z|>2|w|} |K(z-w)-K(z)|\,dz \leqslant A, \tag{1.5}$$

$$|K(z)| \leqslant B\,|z|^{-2n}. \tag{1.6}$$

Then $\mathbf{K}f = K \times f$ *is a bounded operator from* $\mathscr{H}^1$ *to* L^1.

The next section contains the proofs of the following implications of Theorem A: (i) ⇒ (iv) (Theorem 2.2, Lemmas 2.3 and 2.4); (iv) ⇒ (ii) (Proposition 2.5). The equivalence of (iii), (iv) and (vi) is proved in Section 3. In the last section we prove the equivalence of (v) (Theorem 4.6).

Theorem B is discussed at the end of Section 2, while Theorems C and D are the main subject of the last section.

2. Maximal Operators, Atomic Decomposition and $\mathscr{BMO}$

For a tempered distribution f on $\mathbf{C}^n$ the following conditions are equivalent and define the local Hardy space h^1 [5]:

(I_1) $f^*_\sigma(z) = \sup_{\varphi\in\mathscr{B}} \sup_{0<t<\sigma} |\varphi_t * f(z)| \in L^1$;

(I_2) for some $\varphi \in \mathscr{B}$ with $\int \varphi(z)\,dz \neq 0$ $\sup_{0<t<\sigma} |\varphi_t * f(z)| \in L^1$;

(I_3) $f(z) = \sum_j \lambda_j a_j(z)$, where $\sum_j |\lambda_j| < +\infty$, $\operatorname{supp} a_j \subset Q(z_j, r_j)$, $\|a_j\|_\infty \leqslant (2r_j)^{-2n}$ and $\int a_j(z)\,dz = 0$ whenever $r_j < \sigma$;

(I_4) $f \in L^1$ and $R_j * f$, $\bar{R}_j * f \in L^1$, $j = 1,\ldots, n$.

In these statements σ is any fixed positive constant.

LEMMA 2.1. *Assume that* $\operatorname{supp} f \subset Q(z_0, \sigma)$. *Then there is a positive constant* $c(\sigma)$, *independent of* z_0, *such that*

$$c(\sigma)^{-1}\,\|M_\sigma f(z)\|_1 \leqslant \|f(z)\bar{\omega}(z_0, z)\|_{h^1} \leqslant c(\sigma)\,\|M_\sigma f(z)\|_1.$$

Proof. Via a twisted translation, we can reduce to the case $z_0 = 0$. Let $\varphi \in \mathscr{B}$ and $0 < t < \sigma$. Then

$$\begin{aligned}\varphi_t * f(z) &= \int \varphi_t(z-w)\,\bar{\omega}(z, z-w)\,f(w)\,\bar{\omega}(z, w)\,dw \\ &= (\psi^z)_t \times f(z),\end{aligned}$$

where $\psi^z(u) = \varphi(u)\,\bar{\omega}(z, tu)$. We only need estimates for $|z| < 2\sigma$, because all the functions involved vanish otherwise. Then $(1+\sigma^2)^{-1}\psi^z \in \mathscr{B}$ and therefore

$$f_\sigma^*(z) \leqslant (1+\sigma^2)\, M_\sigma f(z).$$

This proves one of the implications. The other implication is proved similarly.

Consider a partition of $\mathbf{C}^n$ into a mesh of cubes $Q_j = Q(z_j, \sigma/2)$ and construct a C^∞ partition of unity $\{\varphi_j\}$ such that φ_j is supported on $Q_j^* = Q(z_j, \sigma)$ and $|\nabla\varphi_j(z)| \leqslant 2\sigma$.

THEOREM 2.2. *Assume $M_\sigma f \in L^1$. Then $g_j(z) = f(z)\,\varphi_j(z)\,\bar{\omega}(z_j, z)$ is in h^1 and $\|M_\sigma f\|_1$ is equivalent to $\sum_j \| g_j\|_{h^1}$.*

Proof. By sublinearity of M_σ, $M_\sigma f(z) \leqslant \sum_j M_\sigma(f\varphi_j)(z)$; hence $\|M_\sigma f\|_1 \leqslant \sum_j \|M_\sigma(f\varphi_j)\|_1$. It follows by Lemma 2.1 that $\|M_\sigma f\|_1 \leqslant c(\sigma) \sum_j \| g_j\|_{h^1}$.

To prove the converse, observe that

$$M_\sigma(f\varphi_j)(z) \leqslant c'(\sigma)\, M_\sigma f(z).$$

Indeed for $\varphi \in \mathscr{B}$

$$\varphi_t \times (f\varphi_j)(z) = (\psi^z)_t \times f(z),$$

where $\psi^z(u) = \varphi(u)\,\varphi_j(z - tu)$ and $(1+\sigma^2)^{-1}\psi^z \in \mathscr{B}$.

By Lemma 2.1 this shows that if $M_\sigma f \in L^1$, then $g_j \in h^1$. Also,

$$\begin{aligned}\sum_j \| g_j\|_{h^1} &\leqslant c''(\sigma) \sum_j \|M_\sigma(f\varphi_j)\|_1 \\ &\leqslant k(\sigma) \sum_j \int M_\sigma f(z)\,\chi_j(z)\,dz,\end{aligned}$$

where χ_j is the characteristic function of the support of $M_\sigma(f\varphi_j)$. Since any point in $\mathbf{C}^n$ belongs to at most 4^{2n} of such supports,

$$\sum_j \| g_j\|_{h^1} \leqslant 4^{2n} k(\sigma)\, \|M_\sigma f\|_1.$$

Remark. Theorem 2.2 shows that the definition of $\mathscr{H}^1$ as the space of distributions f such that $M_\sigma f \in L^1$ does not depend on σ as long as σ is finite. This follows from the same statement for h^1.

As a consequence of Theorem 2.2 and the atomic structure of h^1, every function f in $\mathscr{H}^1$ can be written as $f(z) = \sum_j \lambda_j a_j(z)$, where

(II_1) $\sum_j |\lambda_j| \leqslant C \|f\|_{\mathscr{H}^1}$,

(II_2) a_j is supported in $Q(z_j, r_j)$ and $\|a_j\|_\infty \leqslant (2r_j)^{-2n}$,

(II_3) whenever $r_j < \sigma$, there exists η_j such that $|\eta_j - z_j| < 2\sigma$ and $\int a_j(z)\, \bar{\omega}(\eta_j, z)\, dz = 0$.

Conversely, given a sequence $\{a_j\}$ of functions satisfying (II_1), (II_2), (II_3) and a sequence $\{\lambda_j\}$ such that $\sum_j |\lambda_j| < \infty$, the function $f(z) = \sum_j \lambda_j a_j(z)$ is in $\mathscr{H}^1$ and $\|f\|_{\mathscr{H}^1} \leqslant c \sum_j |\lambda_j|$.

This is not yet the atomic decomposition for $\mathscr{H}^1$ as stated in Theorem A(iv). In order to obtain it, we must first replace condition (II_3) with a "centered" cancellation property.

LEMMA 2.3. *Let a be a function supported on $Q = Q(z_0, r)$, $r < \sigma$, such that*

(i) $\|a\|_\infty < (2r)^{-2n}$,

(ii) $\int a(z)\, \bar{\omega}(\eta, z)\, dz = 0$ *for some* η, $|\eta - z_0| < 2\sigma$. *If σ is sufficiently small, a can be decomposed as $a(z) = \sum_j \lambda_j \alpha_j(z)$, where*

(III_1) $\sum_j |\lambda_j| \leqslant C$,

(III_2) $\operatorname{supp} \alpha_j \subset Q(z_j, r_j)$, $\|\alpha_j\|_\infty \leqslant (2r_j)^{-2n}$,

(III_3) $\int \alpha_j(z)\, \bar{\omega}(z_j, z)\, dz = 0$, *whenever* $r_j < \sigma$.

Proof. Write $a(z) = g^{(1)}(z) + b^{(1)}(z)$, where

$$b^{(1)}(z) = \left[\frac{1}{|Q|} \int_Q a(w)\, \bar{\omega}(z_0, w)\, dw\right] \chi_Q(z)\, \omega(z_0, z).$$

The function $\frac{1}{2} g^{(1)}$ satisfies (III_2) and (III_3).

On the other hand

$$\begin{aligned} |b^{(1)}(z)| &= \frac{1}{|Q|} \left| \int_Q a(w)(\bar{\omega}(z_0, w - z_0) - \bar{\omega}(\eta, w - z_0))\, dw \right| \\ &\leqslant |z_0 - \eta|\, r^{1-2n} \leqslant 2\sigma r^{1-2n}. \end{aligned}$$

Hence for $p = 2n/(2n-1)$, $\|b^{(1)}\|_p \leqslant C\sigma$.

Let $\|f\|_{h^1}$ denote the atomic norm of f in h^1, i.e., $\inf\{\sum_j |\mu_j| : f = \sum_j \mu_j a_j,\ a_j$ h^1-atoms as in (I_3)$\}$.

Observe that $\|f\|_{h^1} \leqslant C(\sigma) \|f_1^*\|_1$, where $C(\sigma)$ is an increasing function of σ. Since $\operatorname{supp} b^{(1)} \subset Q(z_0, \sigma)$, $\bar{\omega}(z_0, z)\, b^{(1)}(z) \in h^1$ and

$$\begin{aligned} \|\bar{\omega}(z_0, z)\, b^{(1)}(z)\|_{h^1} &\leqslant C(\sigma) \|(\bar{\omega}(z_0, z)\, b^{(1)}(z))_1^*\|_1 \\ &\leqslant cC(\sigma)(\sigma + 1)^{2n+1} \|\bar{\omega}(z_0, z)\, b^{(1)}(z)\|_p \\ &\leqslant cC(\sigma)\, \sigma(\sigma + 1)^{2n+1}. \end{aligned}$$

Let σ be small enough to have $cC(\sigma)\,\sigma(\sigma+1)^{2n+1}<\frac{1}{2}$. By the definition of the atomic norm in h^1, we have

$$b^{(1)}(z)=\sum_j v_j^{(1)}a_j^{(1)}(z),$$

where $\sum_j |v_j^{(1)}|<\frac{1}{2}$ and the functions $a_j^{(1)}$ are as in (II_2) and (II_3).

We can now decompose the functions $a_j^{(1)}$ whose support is contained in a cube $Q(z_j, r_j)$ with $r_j<\sigma$, as we did for a. Thus

$$b^{(1)}(z)=g^{(2)}(z)+b^{(2)}(z),$$

where

$$g^{(2)}(z)=\sum_j \lambda_j^{(2)}a_j^{(2)}(z),$$

$a_j^{(2)}$ satisfy (III_2) and (III_3), and $\sum_j |\lambda_j^{(2)}|<1$.

Moreover

$$b^{(2)}(z)=\sum_j v_j^{(2)}a_j^{(2)}(z),$$

where the $a_j^{(2)}$ are as in (II_2) and (II_3) and $\sum_j |v_j^{(2)}|<\frac{1}{4}$.

By iteration we can construct sequences $\{b^{(k)}\}$ and $\{g^{(k)}\}$ such that

$$b^{(k)}=g^{(k+1)}+b^{(k+1)},$$

$g^{(k+1)}(z)=\sum_j \lambda_j^{(k+1)}a_j^{(k+1)}(z)$, where the $a_j^{(k+1)}$ satisfy (III_2), (III_3) and also $\sum_j |\lambda_j^{(k+1)}|<2^{-k+1}$, $b^{(k+1)}(z)=\sum_j v_j^{(k+1)}a_j^{(k+1)}(z)$, where the $a_j^{(k+1)}$ satisfy (II_2), (II_3) and also $\sum_j |v_j^{(k+1)}|<2^{-k-1}$.

This shows that $a(z)=\sum_k g^{(k)}(z)$ and the proof is complete.

In the following choose σ small enough so that Lemma 2.3 holds.

LEMMA 2.4. *Let a be a function supported on a cube $Q(z_0, r)$, $r\geqslant\sigma$, such that $\|a\|_\infty<(2r)^{-2n}$. Then a can be decomposed as $\sum_j \lambda_j a_j(z)$, where the a_j's are atoms as in Theorem* A(iv) *and $\sum_j |\lambda_j|<C$.*

Proof. Assume first that $r\leqslant\sqrt{\pi}$. Without loss of generality we can assume that $z_0=0$. Write $a(z)=a_1(z)+a_2(z)$, where

$$a_1(z)=(4\pi)^{-n}\left(\int a(w)\,dw\right)\chi_{Q(0,\sqrt{\pi})}(z).$$

Then a_2 has mean value zero, is supported on $Q(0,\sqrt{\pi})$ and $\|a_2\|_\infty<(2r)^{-2n}+(4\pi)^{-n}\leqslant C(2\sqrt{\pi})^{-2n}$, since $r\geqslant\sigma$.

On the other hand

$$\int a_1(z)\,\bar{\omega}(\eta, z)\,dz = \hat{a}_1(-i\eta)$$

$$= (4\pi)^{-n}\left(\int a(w)\,dw\right)\hat{\chi}_{Q(0,\sqrt{\pi})}(-i\eta)$$

which vanishes for instance when $\eta = (\sqrt{\pi}, 0,..., 0) = \eta_0$.

We can then regard a_1 as a function supported on $Q(\eta_0, 2\sqrt{\pi})$. Since $\|a_1\|_\infty \leqslant (2\sqrt{\pi})^{-2n}$, $2^{-2n}a_1$ is an $\mathscr{H}^1$-atom.

If $r > \sqrt{\pi}$ we partition $Q(z_0, r)$ by repeated bisection into equal cubes with half-side between $\sqrt{\pi}/2$ and $\sqrt{\pi}$. Correspondingly a decomposes as a convex combination of functions to which the first part of the proof applies.

We have thus proved the implication (i) $\Rightarrow$ (iv) of Theorem A. The converse is an almost immediate consequence of Lemma 2.1.

We prove now the implication (iv) $\Rightarrow$ (ii).

PROPOSITION 2.5. *The maximal operator M_∞ maps $\mathscr{H}^1$-atoms into a ball of L^1.*

Proof. We can assume that a is an atom supported on $Q(0, r)$.

Let $\varphi \in \mathscr{B}$, $t > 0$. Then

$$\varphi_t \times a(z) = \int [\varphi_t(z - w) - \varphi_t(z)]\,a(w)\,\bar{\omega}(z, w)\,dw$$

$$+ \varphi_t(z)\,\hat{a}(-iz).$$

If $|z| > 2r$ and $\varphi_t \times a(z) \neq 0$ we must have $t > |z| - r > \frac{1}{2}|z|$, so that

$$|\varphi_t \times a(z)| \leqslant \frac{2r}{|z|^{2n+1}} + \frac{|\hat{a}(-iz)|}{|z|^{2n}}.$$

Hence

$$\int_{|z|>2r} M_\infty a(z)\,dz \leqslant Cr\int_{|z|>2r} |z|^{-2n-1}\,dz$$

$$+ \int_{|z|>2r} |\hat{a}(-iz)|\,|z|^{-2n}\,dz$$

$$\leqslant C'$$

by Hardy's inequality.

On the other hand

$$\int_{|z|\leqslant 2r} M_\infty a(z)\,dz \leqslant \|a\|_\infty (4r)^{2n} \leqslant 2^{2n}$$

because $M_\infty a(z) \leqslant \|a\|_\infty$.

We conclude this section with a few comments on Theorem B. Part (1) of the statement follows from the atomic description of $\mathscr{H}^1$ given in Theorem A(iv), by the method of [2, p. 593]. Part (2) also can be proved with the same method, since the following families of functions are uniformly bounded in $\mathscr{H}^1$:

(1) functions h such that $\operatorname{supp} h \subset Q(z, r)$, $r < 2\sqrt{\pi}$, $\|h\|_\infty \leqslant (2r)^{-2n}$, $\int h(w)\,\bar{\omega}(\eta, w)\,dw = 0$ for some η, $|\eta - z| < 2\sqrt{\pi}$;

(2) functions k such that $\operatorname{supp} k \subset Q(z, r)$, $r \geqslant r_0$, $\|k\|_\infty \leqslant (2r)^{-2n}$.

3. Connection with the Heisenberg Group

Let $\mathbf{H}_n$ be the $2n+1$-dimensional Heisenberg group, i.e., $\mathbf{C}^n \times \mathbf{R}$ with the composition law

$$(z, s)(z', s') = (z + z', s + s' + \operatorname{Im}(z \cdot \bar{z}')).$$

Denote by $\mathbf{K}_n$ the quotient of $\mathbf{H}_n$ modulo the discrete central subgroup generated by $(0, 2\pi)$. The composition law for $\mathbf{K}_n$ is given by

$$(z, e^{i\theta})(w, e^{i\varphi}) = (z + w, e^{i(\theta+\varphi)}\bar{\omega}(z, w)).$$

If f is a function on $\mathbf{C}^n$, denote by $f^\#$ the function on $\mathbf{K}_n$ defined by $f^\#(z, e^{i\theta}) = f(z)\,e^{-i\theta}$. It is easy to check that $f^\# * g^\# = (f \times g)^\#$.

The Hardy space $H^1(\mathbf{H}_n)$ is defined in terms of maximal operators [4] or equivalently as an atomic space [2]. The same can be done on the quotient group $\mathbf{K}_n$. To be precise, given a function φ in the Schwartz space $\mathscr{S}(\mathbf{H}_n)$, denote by $\varphi_t(z, s)$ the function $t^{-2(n+1)}\varphi(z/t, s/t^2)$. Assume that φ is radial, that is, φ depends only on $|z|$ and s, and that $\int \varphi(z, s)\,dz\,ds \neq 0$. Then the space $H^1(\mathbf{K}_n)$ is the space of all distributions f on $\mathbf{K}_n$ such that

$$\sup_{t>0} |\varphi_t * f(z)| \in L^1(\mathbf{K}_n).$$

The "convolution" $\varphi_t * f$ is defined as a function on $\mathbf{K}_n$ by the left action of $\mathbf{H}_n$ on $\mathbf{K}_n$.

$H^1(\mathbf{K}_n)$ can also be defined as the atomic space related to the homogeneous structure, in the sense of [2], induced on $\mathbf{K}_n$ by its Haar measure and the right invariant pseudo-distance generated by the gauge

$$|(z, e^{i\theta})| = \sup\{|\operatorname{Re} z_j|, |\operatorname{Im} z_j|, |1 - e^{i\theta}|^{1/2}, j = 1,..., n\}.$$

(See [8].)

THEOREM 3.1. *A function f belongs to* $\mathscr{H}^1$ *if and only if* $f^{\#} \in H^1(\mathbf{K}_n)$.

Proof. We show that the mapping P defined on a function $\varphi \in L^1(\mathbf{K}_n)$ by

$$P\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z, e^{i\theta})\, e^{i\theta}\, d\theta$$

is bounded from $H^1(\mathbf{K}_n)$ into $\mathscr{H}^1$.

Let a be an atom for $H^1(\mathbf{K}_n)$. Since this space is invariant under right translations, we can assume that a is supported on $Q^{\#}(0, r) = Q(0, r) \times \{\theta : |1 - e^{i\theta}| < r^2\}$, $Q(0, r)$ being a cube in $\mathbf{C}^n$. Moreover, $\|a\|_\infty \leqslant |Q^{\#}(0, r)|^{-1}$ and $\int a(z, e^{i\theta})\, dz\, d\theta = 0$.

If $r > \sqrt{2}$, $Q^{\#}(0, r) = Q(0, r) \times \mathbf{T}$, so that $|Q^{\#}(0, r)| = (2r)^{2n}$. Hence Pa is supported on $Q(0, r)$ and $\|Pa\|_\infty \leqslant (2r)^{-2n}$. By the concluding remarks of Section 2, $\|Pa\|_{\mathscr{H}^1} \leqslant C$.

If $r \leqslant \sqrt{2}$, $|Q^{\#}(0, r)| \sim r^{2n+2}$. In this case Pa is also supported on $Q(0, r)$ and

$$\begin{aligned} |Pa(z)| &= \left| \frac{1}{2\pi} \int_{|1 - e^{i\theta}| < r^2} a(z, e^{i\theta})\, e^{i\theta}\, d\theta \right| \\ &\leqslant C r^{-2n}. \end{aligned}$$

Decompose Pa as $(Pa)_1 + (Pa)_2$, where

$$(Pa)_1(z) = \left[\frac{1}{|Q(0, r)|} \int_{Q(0,r)} Pa(w)\, dw \right] \chi_{Q(0,r)}(z).$$

The function $\frac{1}{2}(Pa)_2$ is an atom for $\mathscr{H}^1$, while $\|(Pa)_1\|_{n/(n-1)} \leqslant C$. By Lemma 2.1, $\|(Pa)_1\|_{\mathscr{H}^1} \leqslant C'$.

Conversely, let a be an atom in $\mathscr{H}^1$. Here again we can assume that a is supported on $Q(0, r)$. If $r \geqslant \sqrt{2}$, $a^{\#}$ is an atom in $H^1(\mathbf{K}_n)$. If $r < \sqrt{2}$, $a^{\#}$ is supported on $Q(0, r) \times \mathbf{T}$ and can be decomposed into a convex combination of atoms, by splitting $\mathbf{T}$ into subintervals whose length is of the order of r^2.

This identification of $\mathscr{H}^1$ with a closed subspace of $H^1(\mathbf{K}_n)$ allows us to derive results on the former from corresponding results on the latter. Notice that a similar embedding can be used to study $h^1(\mathbf{R}^n)$ as a closed subspace of $H^1(\mathbf{R}^n \times \mathbf{T})$.

By this method we obtain, for instance, that condition (iii) of Theorem A also defines $\mathscr{H}^1$.

THEOREM 3.2. *Let* $\varphi \in \mathscr{S}(\mathbf{C}^n)$ *be a radial function such that* $\int \varphi(z)\,dz \neq 0$. *Then* $f \in \mathscr{H}^1$ *if and only if*

$$\sup_{0<t<1} |\varphi_t \times f(z)| \in L^1.$$

Proof. Let $\eta \in \mathscr{S}(\mathbf{R})$ be a function such that supp $\hat{\eta} \subset [-1, 1]$, $0 \leqslant \hat{\eta} \leqslant 1$, $\hat{\eta}(0) = 1$. Let $\psi(z, s) = \varphi(z)\,\eta(s)$. Then

$$\psi_t * f^{\#}(z, e^{i\theta}) = (\varphi_t \times f)(z)\,\hat{\eta}(t^2)\,e^{-i\theta}$$

Therefore

$$\sup_{t>0} |\psi_t * f^{\#}(z, e^{i\theta})| \leqslant \sup_{0<t<1} |\varphi_t \times f(z)|.$$

If $\sup_{0<t<1} |\varphi_t \times f(z)| \in L^1$, then $f^{\#} \in H^1(\mathbf{K}_n)$ and by Theorem 3.1, $f \in \mathscr{H}^1$.

Replacing η by a function $\delta \in \mathscr{S}(\mathbf{R})$ such that $\hat{\delta} \equiv 1$ on $[-1, 1]$ we obtain the converse.

Remark. A slight modification of this result shows that $\mathscr{H}^1$ can be also defined in terms of the maximal operator related to the fundamental solution

$$q_t(z) = \pi^{-(n+1)}[\sinh(2t)]^{-n} \exp[-|z|^2/tgh(2t)]$$

of the "heat equation" $\partial_t q_t = L q_t$ [11].

4. SINGULAR INTEGRALS

THEOREM 4.1. *Let K be a distribution which is locally integrable away from the origin. If*

$$\int_{|z|>2|w|} |K(z-w)\,\bar{\omega}(z, w) - K(z)|\,dz \leqslant A \tag{4.1}$$

and

$$\|K \times f\|_2 \leqslant B\,\|f\|_2 \tag{4.2}$$

Then the operator $\mathbf{K}f = K \times f$ *is bounded from* $\mathscr{H}^1$ *to* L^1.

Proof. It is enough to prove that $\mathbf{K}$ maps all atoms into a bounded subset of L^1. Let a be an atom. We may assume that a is supported on $Q(0, r)$, $r < 2\sqrt{\pi}$. Then, since $\int a(w)\,dw = 0$, for $|z| > 2r$

$$\mathbf{K}a(z) = \int [K(z-w)\,\bar{\omega}(z,w) - K(z)]\,a(w)\,dw.$$

Thus

$$\int_{|z|>2r} |\mathbf{K}a(z)|\,dz \leqslant \int_{|z|>2r} \int |K(z-w)\,\bar{\omega}(z,w) - K(z)|\,|a(w)|\,dw\,dz$$
$$\leqslant A.$$

On the other hand

$$\int_{|z|\leqslant 2r} |\mathbf{K}a(z)|\,dz \leqslant \|\mathbf{K}a\|_2 \left(\int_{|z|\leqslant 2r} dz\right)^{1/2}$$
$$\leqslant CB\,\|a\|_2 (2r)^n \leqslant CB.$$

LEMMA 4.2. *Let K be a compactly supported distribution on $\mathbf{R}^n$ locally integrable away from the origin, and let $\psi\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{T}$ be a measurable function. Suppose that*

$$\int_{|x|>2|y|} |K(x-y)\,\psi(x,y) - K(x)|\,dx \leqslant A$$

for any $y \neq 0$. Then $|x|\,K(x) \in L^1(\mathbf{R}^n)$.

Proof. Let us deal first with the case $n = 1$. If $2/3 < a < 1$, then $|a^m - a^{m-1}| < \frac{1}{2}a^m$, so that

$$\int_{a^m}^{\infty} |K(x)|\,dx \leqslant \int_{a^m}^{\infty} |K(x - a^m + a^{m-1})\,\psi(x, a^m - a^{m-1}) - K(x)|\,dx$$
$$+ \int_{a^m}^{\infty} |K(x - a^m + a^{m-1})|\,dx \leqslant A + \int_{a^{m-1}}^{\infty} |K(x)|\,dx.$$

By the same argument,

$$\int_{|x|>a^m} |K(x)|\,dx \leqslant 2A + \int_{|x|>a^{m-1}} |K(x)|\,dx.$$

Therefore there is a constant B depending only on the support of K such that

$$\int_{|x|>a^m} |K(x)|\,dx \leqslant BAm,$$

which implies that $|x|\,K(x)\in L^1(\mathbf{R})$. In dimension n larger than one, a similar inequality is obtained by means of an n-fold iteration of the argument above, with translations in orthogonal directions.

COROLLARY 4.3. *Let K be a compactly supported distribution locally integrable away from the origin in $\mathbf{C}^n$. Then*

$$\int_{|z|>2|w|} |K(z-w)\,\bar{\omega}(z,w)-K(z)|\,dz<\infty$$

if and only if

$$\int_{|z|>2|w|} |K(z-w)-K(z)|\,dz<\infty.$$

Proof. The statement follows from Lemma 4.2 and the fact that K has compact support, because

$$\begin{aligned}
&\int_{|z|>2|w|} |K(z-w)\,\bar{\omega}(z,w)-K(z)|\,dz\\
&\quad\leqslant\int_{|z|>2|w|} |K(z-w)-K(z)|\,dz+\int_{|z|>2|w|} |K(z)|\,|\bar{\omega}(z,w)-1|\,dz\\
&\quad\leqslant\int_{|z|>2|w|} |K(z-w)-K(z)|\,dz+\int_{|z|>2|w|} |w|\,|z|\,|K(z)|\,dz
\end{aligned}$$

and conversely,

$$\begin{aligned}
&\int_{|z|>2|w|} |K(z-w)-K(z)|\,dz\\
&\quad\leqslant\int_{|z|>2|w|} |K(z-w)\,\bar{\omega}(z,w)-K(z)|\,dz+\int_{|z|>2|w|} |w|\,|z|\,|K(z)|\,dz.
\end{aligned}$$

THEOREM 4.4. *Let K be a compactly supported distribution which is locally integrable away from the origin. Suppose that* (1.3) *holds and that $|\hat{K}(\zeta)|<B$. Then the operator $\mathbf{K}f=K\times f$ is bounded from $\mathscr{H}^1$ to L^1.*

Proof. In view of Corollary 4.3, it is enough to modify the part of the proof of Theorem 4.1 concerning the estimate for $|z|<2r$. One has:

$$\begin{aligned}
\int_{|z|\leqslant 2r} |\mathbf{K}a(z)|\,dz
&\leqslant \int_{|z|\leqslant 2r} |K * a(z)|\,dz + \int_{|z|\leqslant 2r}\int |K(z-w)|\,|\bar{\omega}(z,w)-1|\,|a(w)|\,dw\,dz\\
&\leqslant CB + \int |w|\,|a(w)| \int_{|z|\leqslant 2r} |z-w|\,|K(z-w)|\,dz\,dw\\
&\leqslant CB + 2\sqrt{\pi}\,\|\,|z|\,K(z)\|_1.
\end{aligned}$$

Piecing together these results, one has:

COROLLARY 4.5. *If K satisfies the conditions of Theorem C, then the operator $\mathbf{K}f = K \times f$ is bounded from $\mathscr{H}^1$ to L^1.*

The next result completes the proof of Theorem B:

THEOREM 4.6. *A function f belongs to $\mathscr{H}^1$ if and only if f, $R_j \times f$, $\bar{R}_j \times f$ are in L^1, $j = 1,\dots, n$.*

We need two lemmas.

LEMMA 4.7. *Let φ be a Lipschitz function on $\mathbf{C}^n$ and denote by M_φ the operator of multiplication by φ. Then the commutators $[\mathbf{R}_j, M_\varphi]$, $[\bar{\mathbf{R}}_j, M_\varphi]$, $j = 1,\dots, n$, are bounded on L^1.*

Proof. Let f be in L^1. Then

$$\begin{aligned}
&R_j \times (\varphi f)(z) - \varphi(z)(R_j \times f)(z)\\
&\quad = \int R_j(z-w)[\varphi(w)-\varphi(z)]\,f(w)\,\bar{\omega}(z,w)\,dw
\end{aligned}$$

so that

$$\begin{aligned}
\|[\mathbf{R}_j, M_\varphi]\,f\|_1 &\leqslant C \int dz \int |R_j(z-w)|\,|z-w|\,|f(w)|\,dw\\
&\leqslant C'\,\|f\|_1
\end{aligned}$$

because $|z|\,R_j(z) \in L^1$.

LEMMA 4.8. *Assume that f is an L^1 function supported on $Q(z_0, 1)$. Then*

$$\|R_j \times f - \omega(z_0,\cdot)\,R_j * (f\bar{\omega}(z_0,\cdot))\|_1 \leqslant C\,\|f\|_1$$

and similarly for $\bar{R}_j$, $j = 1,\dots, n$.

Proof. The result follows easily from the estimate:

$$
\begin{aligned}
&|R_j \times f(z) - \omega(z_0, z)\, R_j * (f\bar{\omega}(z_0, \cdot))(z)| \\
&\qquad \leqslant \int |R_j(z-w)|\, |f(w)|\, |\bar{\omega}(z, w) - \bar{\omega}(z_0, w-z)|\, dw \\
&\qquad \leqslant c \int |R_j(z-w)|\, |z-w|\, |f(w)|\, dw
\end{aligned}
$$

because R_j has compact support.

Proof of Theorem 4.6. Let $\{\varphi_k\}$ be the partition of unity used in the proof of Theorem 2.2 (corresponding to $\sigma = 1$). Then, assuming that f, $R_j \times f$, $\bar{R}_j \times f \in L^1$, we have

$$
\begin{aligned}
\|f\|_{\mathscr{H}^1} &\leqslant \sum_k \|f\varphi_k\|_{\mathscr{H}^1} \\
&\leqslant c_1 \sum_k \|f\varphi_k \bar{\omega}(z_k, \cdot)\|_{h^1} \\
&\leqslant c_2 \sum_k \Big[\|f\varphi_k \bar{\omega}(z_k, \cdot)\|_1 \\
&\qquad + \sum_{1 \leqslant j \leqslant n} (\|R_j * (f\varphi_k \bar{\omega}(z_k, \cdot))\|_1 \\
&\qquad + \|\bar{R}_j * (f\varphi_k \bar{\omega}(z_k, \cdot))\|_1) \Big] \\
&\leqslant c_3 \sum_k \Big[\|f\varphi_k\|_1 + \sum_{1 \leqslant j \leqslant n} (\|R_j \times (f\varphi_k)\|_1 + \|\bar{R}_j \times (f\varphi_k)\|_1) \Big]
\end{aligned}
$$

by Lemma 4.8. Now applying Lemma 4.7, we have

$$
\begin{aligned}
\|f\|_{\mathscr{H}^1} &\leqslant c_4 \sum_k \Big[\|f\varphi_k\|_1 + \sum_{1 \leqslant j \leqslant n} (\|(R_j \times f)\, \varphi_k\|_1 + \|(\bar{R}_j \times f)\, \varphi_k\|_1) \Big] \\
&= c_4 \Big[\|f\|_1 + \sum_{1 \leqslant j \leqslant n} (\|R_j \times f\|_1 + \|\bar{R}_j \times f\|_1) \Big].
\end{aligned}
$$

In order to complete the proof of Theorem C, we need to show first that the Riesz transforms map $\mathscr{H}^1$ into $\mathscr{H}^1$.

LEMMA 4.9. *The Riesz transforms are bounded on* $\mathscr{H}^1$.

Proof. Consider on the Heisenberg group $\mathbf{H}_n$ the operators defined by left convolution with the kernels

$$S_j(z,t)=\frac{z_j}{(|z|^4+t^2)^{(2n+3)/4}},\qquad \bar{S}_j(z,t)=\frac{\bar{z}_j}{(|z|^4+t^2)^{(2n+3)/4}}.$$

A result of Knapp and Stein [7] asserts that such operators are bounded on $L^2(\mathbf{H}_n)$. If $(|z|^4+t^2)^{1/4}>2(|w|^4+s^2)^{1/4}$ they also satisfy

$$|S_j((z,t)(w,s)^{-1})-S_j((z,t))|\leqslant C\frac{(|w|^4+s^2)^{1/4}}{(|z|^4+t^2)^{(2n+3)/4}}$$

Hence the kernels T_j and $\bar{T}_j$ on $\mathbf{K}_n$ obtained periodizing S_j and $\bar{S}_j$ with respect to the central variable t are bounded on $L^2(\mathbf{K}_n)$ and satisfy, for $|(z,e^{i\theta})|>2\,|(w,e^{i\varphi})|$

$$|T_j((z,e^{i\theta})(w,e^{i\varphi})^{-1})-T_j((z,e^{i\theta}))|\leqslant C\frac{|(w,e^{i\varphi})|}{|(z,e^{i\theta})|^{2n+3}}.$$

It follows from [2, p. 599] that they map $H^1(\mathbf{K}_n)$ boundedly into itself.
For a function in $H^1(\mathbf{K}_n)$ of the form $f^{\#}, f\in\mathscr{H}^1$,

$$(T_j*f^{\#})(z,e^{i\theta})=(\hat{T_j}(1)\times f)^{\#},$$

where

$$\begin{aligned}\hat{T_j}(1)&=\frac{1}{2\pi}\int_0^{2\pi}T_j(z,e^{i\theta})\,e^{-i\theta}\,d\theta\\&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_j(z,t)\,e^{-it}\,dt\\&=\frac{1}{2\pi}\frac{z_j}{|z|^{2n+1}}\int_{-\infty}^{+\infty}(1+s^2)^{-(2n+3)/4}\exp(-is\,|z|^2)\,ds\\&=\frac{z_j}{|z|^{2n+1}}\,\psi(|z|^2),\end{aligned}$$

where ψ is a continuous rapidly decreasing function [13, p. 132].

Hence $\hat{T_j}(1)$ differs from R_j by an L^1 function. Therefore $\mathbf{R}_j$ is bounded on $\mathscr{H}^1$ by Theorem 3.1.

We can now complete the proof of Theorem C.

Proof of Theorem C. Let $\mathbf{K}$ be a kernel satisfying the hypotheses of Theorem C. By Corollary 4.5 we already know that $\mathbf{K}$ maps $\mathscr{H}^1$ into L^1. By

Theorem 4.6 we only need to show that the same is true for $\mathbf{R}_j\mathbf{K}$, $\bar{\mathbf{R}}_j\mathbf{K}$, $j = 1,..., n$.

But $\mathbf{R}_j\mathbf{K}$ can be written as $\mathbf{K}\mathbf{R}_j + [\mathbf{R}_j, \mathbf{K}]$. By Lemma 4.9, $\mathbf{K}\mathbf{R}_j$ maps $\mathscr{H}^1$ into L^1. Therefore the proof will be complete as soon as we show that $[\mathbf{R}_j, \mathbf{K}]$ is bounded on L^1.

Because of Lemma 4.2, the estimate

$$\begin{aligned} |R_j \times K(z) - K \times R_j((z)| &\leqslant \int |R_j(z-w)|\, |K(w)|\, |\bar{\omega}(z, w) - \omega(z, w)|\, dw \\ &\leqslant \int |R_j(z-w)|\, |K(w)|\, |1 - \omega^2(z, w)|\, dw \\ &\leqslant 2 \int |R_j(z-w)|\, |z-w|\, |K(w)|\, |w|\, dw \end{aligned}$$

shows that $[\mathbf{R}_j, \mathbf{K}]$ is twisted convolution by an L^1 function.

Proof of Theorem D. Let a be an atom supported on $Q(0, r)$, $r < 2\sqrt{\pi}$, such that $\int a(w)\, dw = 0$. For $|z| > 2r$

$$\begin{aligned} K \times a(z) = \int [K(z-w) - K(z)]\, a(w)\, \bar{\omega}(z, w)\, dw \\ + K(z)\, \hat{a}(-iz). \end{aligned}$$

By (1.5) the integral over $|z| > 2r$ of the first term can be estimated as in Theorem 4.1. On the other hand

$$\begin{aligned} \int_{|z| > 2r} |K(z)\, \hat{a}(-iz)|\, dz &\leqslant \int_{|z| > 2r} \frac{|\hat{a}(-iz)|}{|z|^{2n}}\, dz \\ &\leqslant C \end{aligned}$$

by Hardy's inequality. The estimate

$$\int_{|z| < 2r} |\mathbf{K}a(z)|\, dz \leqslant C$$

follows as in the proof of Theorem 4.1, once we remark that K is in L^2 and that L^2 is an algebra for twisted convolution [12].

COROLLARY 4.10. *If K is a distribution which satisfies either the assumptions of Theorem C or those of Theorem D, then the twisted convolution operator* $\mathbf{K}$ *is bounded from L^∞ to $\mathscr{BMO}$.*

Proof. Since a kernel satisfying the assumptions above is bounded from

$\mathscr{H}^1$ to L^1 (Corollary 4.5 and Theorem D), the result follows via Theorem B, because the adjoint operator $\mathbf{K}^*$ is associated with the kernel $K^*(z) = \overline{K(-z)}$, which satisfies the same assumptions.

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